RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, MARCH 2022
SECOND YEAR [BATCH 2020-23]

Date : 07/03/2022	**MATHEMATICS**	
Time : 11am-1pm	**Paper** : MTMA CC7	Full Marks : 50

Group A

Answer any 5 questions. **[5 x 5 = 25 marks]**

1. Show that the function $f : [a, b] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} -1, & x \in [a, b] \cap \mathbb{Q} \\ 1, & x \in [a, b] - \mathbb{Q} \end{cases}$$
is not Riemann integrable on $[a, b]$. [5]

2. Let $f : [0, 1] \to \mathbb{R}$ be a function defined by
$$f(x) = \begin{cases} x^2, & \text{when x is rational} \\ 1, & \text{when x is irrational.} \end{cases}$$
Examine the Riemann integrability of f on $[0, 1]$. [5]

3. Show that $\frac{1}{3\sqrt{2}} < \int\limits_0^1 \frac{x^2}{\sqrt{1+x}} dx < \frac{1}{3}$. [5]

4. (a) Find a point $c \in [0, 1]$ such that $\int\limits_0^1 \frac{x}{1+x^2} dx = c \int\limits_0^1 \frac{dx}{1+x^2} = 1$, by first MVT.

   (b) Show that $\lim\limits_{x\to 0} \frac{1}{x^3} \int\limits_0^{x^2} \sin\sqrt{t}\, dt = \frac{2}{3}$. [3+2]

5. Using second MVT of integral calculus in Weirstrass form show that
$|\int\limits_a^b \frac{\cos x}{1+x} dx| < \frac{4}{1+a}, 0 < a < b.$ [5]

6. Verify the first MVT of integral calculus for the function $f(x) = x^2 \sin x, \quad x \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. [5]

7. Evaluate $\lim\limits_{n\to\infty} [\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}]$ as an integral. [5]

Group B

Answer any 5 questions. **[5 x 5 = 25 marks]**

8. Show that $f(x) = \begin{cases} x^2 \cos\frac{\pi}{x^2}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is not a function of Bounded Variation. [5]

9. For what values of p, the integral $\int\limits_0^\infty \frac{\sin x}{x^p+1} dx$ is absolutely convergent. [5]

10. If $I_n = \int \frac{dx}{(x^2+1)^n}$, prove that $2(n-1)I_n = (2n-3)I_{n-1} + \frac{x}{(x^2+1)^{n-1}}$. [5]

11. Show that the sequence of functions $\{f_n\}_n$ defined by $f_n(x) = x^n(1-x^2), 0 < x \leq 1$, converges uniformly on $(0, 1]$. [5]

12. Show that $\sum\limits_{n=1}^{\infty} xe^{-nx}$ is not uniformly convergent on $[0, 1]$. [5]

13. Determine the interval of uniform convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(x-3)^n}{n.5^n}$. [5]

14. Show that for $\alpha > -1$, $\int\limits_0^{\infty} e^{-x^2} x^\alpha dx$ converges to $\frac{1}{2}\Gamma(\frac{\alpha+1}{2})$.

    Hence deduce that $\int\limits_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. [4+1]

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